Lie bialgebras of generalized Virasoro-like type[1]

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Abstract. In two recent papers by the authors, all Lie bialgebra structures on Lie algebras of generalized Witt type are classified. In this paper all Lie bialgebra structures on generalized Virasoro-like algebras are determined. It is proved that all such Lie bialgebras are triangular coboundary.

Key words: Lie bialgebras, Yang-Baxter equation, generalized Virasoro-like algebras.

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§1. Introduction

The notion of Lie bialgebras was first introduced by Drinfeld in 1983 [D1] (cf. [D2]) in a connection with quantum groups. Since then there appeared a number of papers on Lie bialgebras (e.g., [M1, M2, NT, N, SS, WS, T]). Michaelis [M1] presented a class of Lie bialgebras containing the Virasoro algebra (this type of Lie bialgebras was classified by Ng and Taft [NT], cf. [N, T]) and gave a method on how to obtain the structure of a triangular coboundary Lie bialgebra on a Lie algebra containing two elements a, b satisfying $[a,b]=b$.

In two recent papers [SS, WS], all Lie bialgebra structures on Lie algebras of generalized Witt type are classified. In this paper we shall determine all Lie bialgebra structures on a class of Lie algebras (cf. (1.11)), referred to as the *generalized Virasoro-like algebras* (the structure and representation theories of the Virasoro-like algebra have attracted some authors' attentions because of its close relation with the Virasoro algebra, e.g., [LT, MJ, X2, X3, ZM, ZZ]).

Let us recall the definition of Lie bialgebras. For a vector space $\mathcal{L}$ over the complex field $\mathbb{C}$, we define the *twist map* τ of $\mathcal{L}\otimes\mathcal{L}$ and the *cyclic map* ξ of $\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L}$ by

$$\tau : x\otimes y\mapsto y\otimes x,\qquad \xi : x\otimes y\otimes z\mapsto y\otimes z\otimes x\quad \text{for}\quad x,y,z\in\mathcal{L}. \tag{1.1}$$

Then a *Lie algebra* can be defined as a pair $(\mathcal{L},\varphi)$ consisting of a vector space $\mathcal{L}$ and a bilinear map $\varphi:\mathcal{L}\otimes\mathcal{L}\to\mathcal{L}$ (the *bracket* of $\mathcal{L}$) satisfying the following conditions,

$$\mathrm{Ker}(1-\tau)\subset \mathrm{Ker}\,\varphi\quad \text{(skew-symmetry)}, \tag{1.2}$$

$$\varphi\cdot(1\otimes\varphi)\cdot(1+\xi+\xi^2)=0:\ \mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L}\to\mathcal{L}\quad \text{(Jacobi identity)}, \tag{1.3}$$

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where 1 is the identity map of $\mathcal{L}\otimes\mathcal{L}$. A *Lie coalgebra* is a pair $(\mathcal{L},\Delta)$ consisting of a vector space $\mathcal{L}$ and a linear map $\Delta:\mathcal{L}\to\mathcal{L}\otimes\mathcal{L}$ (*cobracket* of $\mathcal{L}$) satisfying the following conditions:

$$\mathrm{Im}\,\Delta\subset\mathrm{Im}(1-\tau)\quad\text{(anti-commutativity)},\tag{1.4}$$

$$(1+\xi+\xi^2)\cdot(1\otimes\Delta)\cdot\Delta=0:\ \mathcal{L}\to\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L}\quad\text{(Jacobi identity)}.\tag{1.5}$$

Definition 1.1. A *Lie bialgebra* is a triple $(\mathcal{L},\varphi,\Delta)$ such that $(\mathcal{L},\varphi)$ is a Lie algebra and $(\mathcal{L},\Delta)$ is a Lie coalgebra and the following *compatibility condition* holds:

$$\Delta\varphi(x,y)=x\cdot\Delta y-y\cdot\Delta x\quad\text{ for }\quad x,y\in\mathcal{L},\tag{1.6}$$

where the symbol "$\cdot$" means the action

$$x\cdot(\sum_i a_i\otimes b_i)=\sum_i([x,a_i]\otimes b_i+a_i\otimes[x,b_i])\tag{1.7}$$

for $x,a_i,b_i\in\mathcal{L}$, and in general $[x,y]=\varphi(x,y)$ for $x,y\in\mathcal{L}$.

One shall notice that the significant difference between Lie bialgebras and (associative) bialgebras lies in the compatibility condition (1.6): A bialgebra requires that Δ is an algebra morphism: $\Delta\cdot\varphi=(\varphi\otimes\varphi)\cdot(1\otimes\tau\otimes1)\cdot\Delta\otimes\Delta$, while a Lie bialgebra requires that Δ is a derivation (cf. (1.13)) of $\mathcal{L}\to\mathcal{L}\otimes\mathcal{L}$. Thus the properties of Lie bialgebras are not similar to those of bialgebras.

Definition 1.2. (1) A *coboundary Lie bialgebra* is a $(\mathcal{L},\varphi,\Delta,r)$, where $(\mathcal{L},\varphi,\Delta)$ is a Lie bialgebra and $r\in\mathrm{Im}(1-\tau)\subset\mathcal{L}\otimes\mathcal{L}$ such that Δ is a *coboundary of* r, i.e. $\Delta=\Delta_r$, where in general Δ_r (which is an inner derivation, cf. (1.14)) is defined by,

$$\Delta_r(x)=x\cdot r\quad\text{ for }\quad x\in\mathcal{L}.\tag{1.8}$$

(2) A coboundary Lie bialgebra $(\mathcal{L},\varphi,\Delta,r)$ is *triangular* if it satisfies the following *classical Yang-Baxter Equation* (CYBE):

$$c(r)=0,\tag{1.9}$$

where $c(r)$ is defined by

$$c(r)=[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},r^{23}],\tag{1.10}$$

and r^{ij} are defined as follows: Denote $\mathcal{U}(\mathcal{L})$ the universal enveloping algebra of $\mathcal{L}$ and 1 the identity element of $\mathcal{U}(\mathcal{L})$. If $r=\sum_i a_i\otimes b_i\in\mathcal{L}\otimes\mathcal{L}$, then

$$\begin{aligned}
r^{12}&=r\otimes1=\sum_i a_i\otimes b_i\otimes1,\\
r^{13}&=(1\otimes\tau)(\tau\otimes1)=\sum_i a_i\otimes1\otimes b_i,\\
r^{23}&=1\otimes r=\sum_i 1\otimes a_i\otimes b_i,
\end{aligned}$$

are all elements in $\mathcal{U}(\mathcal{L}) \otimes \mathcal{U}(\mathcal{L}) \otimes \mathcal{U}(\mathcal{L})$.

Let us state our main results below. For any *nondegenerate* additive subgroup Γ of $\mathbb{C}^2$ (namely, Γ contains a $\mathbb{C}$-basis of $\mathbb{C}^2$), the *generalized Virasoro-like algebra* $\mathcal{L}(\Gamma)$ is a Lie algebra with basis $\{L_\alpha, \partial_1, \partial_2 \,|\, \alpha \in \Gamma\backslash\{0\}\}$ and bracket

$$[L_\alpha, L_\beta] = (\alpha_1\beta_2 - \beta_1\alpha_2)L_{\alpha+\beta}, \quad [\partial_i, L_\alpha] = \alpha_i L_\alpha \ \text{ for } \ \alpha, \beta \in \Gamma, \ i = 1, 2, \tag{1.11}$$

where we use the convention that if an undefined notation appears in an expression, we always treat it as zero; for instance, $L_\alpha = 0$ if $\alpha = 0$. In particular, when $\Gamma = \mathbb{Z}^2$, the derived subalgebra $[\mathcal{L}(\mathbb{Z}^2), \mathcal{L}(\mathbb{Z}^2)] = \text{span}\{L_\alpha \,|\, \alpha \in \mathbb{Z}^2\backslash\{0\}\}$ is the (centerless) Virasoro-like algebra (e.g., [LT, MJ, ZZ]). The Lie algebra $\mathcal{L}(\Gamma)$ is closely related to the Lie algebras of Block type (cf. [DZ, X3, Z1]) and the Lie algebras of Cartan type S (cf. [SX, X1, Z2]).

For a Lie algebra $\mathcal{L}$ and an $\mathcal{L}$-module V, denote by $H^1(\mathcal{L}, V)$ the *first cohomology group* of $\mathcal{L}$ with coefficients in V. It is well-known that

$$H^1(\mathcal{L}, V) \cong \text{Der}(\mathcal{L}, V)/\text{Inn}(\mathcal{L}, V), \tag{1.12}$$

where $\text{Der}(\mathcal{L}, V)$ is the set of *derivations* $d : \mathcal{L} \to V$ which are linear maps satisfying

$$d([x, y]) = x \cdot d(y) - y \cdot d(x) \ \text{ for } \ x, y \in \mathcal{L}, \tag{1.13}$$

and $\text{Inn}(\mathcal{L}, V)$ is the set of *inner derivations* a_{inn}, $a \in V$, defined by

$$a_{\text{inn}} : x \mapsto x \cdot a \ \text{ for } \ x \in \mathcal{L}. \tag{1.14}$$

An element r in a Lie algebra $\mathcal{L}$ is said to satisfy the *modern Yang-Baxter equation* (MYBE) if

$$x \cdot c(r) = 0 \ \text{ for all } \ x \in \mathcal{L}. \tag{1.15}$$

The main results of this paper is the following.

Theorem 1.3. (1) *Every Lie bialgebra structure on the Lie algebra* $\mathcal{L}(\Gamma)$ *defined in* (1.11) *is a triangular coboundary Lie bialgebra.*

(2) *An element* $r \in \mathcal{L}(\Gamma)$ *satisfies CYBE in* (1.9) *if and only if it satisfies MYBE in* (1.15).

(3) *Regarding* $V = \mathcal{L}(\Gamma) \otimes \mathcal{L}(\Gamma)$ *as an* $\mathcal{L}(\Gamma)$*-module under the adjoint diagonal action of* $\mathcal{L}(\Gamma)$ *in* (1.7), *we have* $H^1(\mathcal{L}(\Gamma), V) = \text{Der}(\mathcal{L}(\Gamma), V)/\text{Inn}(\mathcal{L}(\Gamma), V) = 0$.

§2. Proof of the main results

First we retrieve some useful results from Drinfeld [D2], Michaelis [M1], Ng-Taft [NT] and combine them into the following theorem.

Theorem 2.1. (1) *For a Lie algebra* $\mathcal{L}$ *and* $r \in \mathrm{Im}(1-\tau) \subset \mathcal{L}$*, the tripple* $(\mathcal{L}, [\cdot,\cdot], \Delta_r)$ *is a Lie bialgebra if and only if* r *satisfies MYBE* [D2].

(2) *Let* $\mathcal{L}$ *be a Lie algebra containing two elements* a, b *satisfying* $[a,b] = b$*, and set* $r = a \otimes b - b \otimes a$*. Then* Δ_r *equips* $\mathcal{L}$ *with the structure of a triangular coboundary Lie bialgebra* [M1].

(3) *For a Lie algebra* $\mathcal{L}$ *and* $r \in \mathrm{Im}(1-\tau) \subset \mathcal{L}$*, we have* [NT]

$$(1+\xi+\xi^2)\cdot(1\otimes\Delta)\cdot\Delta(x) = x\cdot c(r) \quad \textit{for all} \quad x \in \mathcal{L}. \tag{2.1}$$

We shall follow [SS, WS] closely to prove Theorem 1.3.

First Theorem 1.3(2) follows from the following more general result.

Lemma 2.2. *Denote by* $\mathcal{L}(\Gamma)^{\otimes n}$ *the tensor product of* n *copies of* $\mathcal{L}(\Gamma)$*. Regarding* $\mathcal{L}(\Gamma)^{\otimes n}$ *as an* $\mathcal{L}(\Gamma)$*-module under the adjoint diagonal action of* $\mathcal{L}(\Gamma)$*, suppose* $c \in \mathcal{L}(\Gamma)^{\otimes n}$ *satisfying* $a \cdot c = 0$ *for all* $a \in \mathcal{L}(\Gamma)$*. Then* $c = 0$*.*

Proof. The lemma is obtained by using the same arguments in the proof of [WS, Lemma 2.2]. □

Theorem 1.3(3) follows from the following proposition.

Proposition 2.3. $\mathrm{Der}(\mathcal{L}(\Gamma), V) = \mathrm{Inn}(\mathcal{L}(\Gamma), V)$*, where* $V = \mathcal{L}(\Gamma)\otimes\mathcal{L}(\Gamma)$*.*

Proof. We shall prove the result by several claims. Note that $V = \oplus_{\alpha\in\Gamma} V_\alpha$ is Γ-graded with $V_\alpha = \sum_{\beta+\gamma=\alpha} \mathcal{L}(\Gamma)_\beta \otimes \mathcal{L}(\Gamma)_\gamma$, where $\mathcal{L}(\Gamma)_\alpha = \mathbb{C}L_\alpha \oplus \delta_{\alpha,0}(\mathbb{C}\partial_1 + \mathbb{C}\partial_2)$ for $\alpha \in \Gamma$. A derivation $D \in \mathrm{Der}(\mathcal{L}(\Gamma), V)$ is *homogeneous of degree* $\alpha \in \Gamma$ if $D(V_\beta) \subset V_{\alpha+\beta}$ for all $\beta \in \Gamma$. Denote $\mathrm{Der}(\mathcal{L}(\Gamma), V)_\alpha = \{D \in \mathrm{Der}(\mathcal{L}(\Gamma), V) \,|\, \deg D = \alpha\}$ for $\alpha \in \Gamma$.

Claim 1. Let $D \in \mathrm{Der}(\mathcal{L}(\Gamma), V)$. Then

$$D = \sum_{\alpha\in\Gamma} D_\alpha, \quad \text{where} \quad D_\alpha \in \mathrm{Der}(\mathcal{L}(\Gamma), V)_\alpha, \tag{2.2}$$

which holds in the sense that for every $u \in \mathcal{L}(\Gamma)$, only finitely many $D_\alpha(u) \neq 0$, and $D(u) = \sum_{\alpha\in\Gamma} D_\alpha(u)$ (we call such a sum in (2.2) *summable*).

For $\alpha \in \Gamma$, we define D_α as follows: For any $u \in \mathcal{L}(\Gamma)_\beta$ with $\beta \in \Gamma$, write $d(u) = \sum_{\gamma\in\Gamma} v_\gamma \in V$ with $v_\gamma \in V_\gamma$, then we set $D_\alpha(u) = v_{\alpha+\beta}$. Obviously $D_\alpha \in \mathrm{Der}(\mathcal{L}(\Gamma), V)_\alpha$ and we have (2.2).

Claim 2. If $\alpha \neq 0$, then $D_\alpha \in \mathrm{Inn}(\mathcal{L}(\Gamma), V)$.

Denote $T = \mathrm{span}\{\partial_1, \partial_2\}$ and define the nondegenerate bilinear map from $\Gamma \times T \to \mathbb{C}$,

$$\partial(\alpha) = \langle \partial, \alpha \rangle = \langle \alpha, \partial \rangle = a_1\alpha_1 + a_2\alpha_2 \text{ for } \alpha = (\alpha_1, \alpha_2) \in \Gamma,\ \partial = a_1\partial_1 + a_2\partial_2 \in T. \tag{2.3}$$

By linear algebra, one can choose $\partial \in T$ with $\partial(\alpha) \neq 0$. Denote $a = (\partial(\alpha))^{-1} D_\alpha(\partial) \in \mathcal{L}(\Gamma)_\alpha$. Then for any $x \in \mathcal{L}(\Gamma)_\beta, \beta \in \Gamma$, applying D_α to $[\partial, x] = \partial(\beta)x$, using $D_\alpha(x) \in V_{\alpha+\beta}$, We have

$$\partial(\alpha+\beta)D_\alpha(x) - x \cdot D_\alpha(\partial) = \partial \cdot D_\alpha(x) - x \cdot D_\alpha(\partial) = \partial(\beta)D_\alpha(x), \tag{2.4}$$

i.e., $D_\alpha(x) = a_{\mathrm{inn}}(x)$. Thus $D_\alpha = a_{\mathrm{inn}}$ is inner.

Claim 3. $D_0 \in \mathrm{Inn}(\mathcal{W}, V)$.

Choose a $\mathbb{C}$-basis $\{\varepsilon_1, \varepsilon_2\} \subset \Gamma$ of $\mathbb{C}$. Define $\partial'_i \in T$ by $\langle \partial'_i, \varepsilon_j \rangle = \delta_{ij}$. Let $\Gamma' = \{(p, q) \in \mathbb{C}^2 \mid p\varepsilon_1 + q\varepsilon_2 \in \Gamma\}$. Then $\mathbb{Z}^2 \subset \Gamma'$. We write $L_{p,q} = L_{p\varepsilon_1 + q\varepsilon_2}$, and re-denote ∂'_i and Γ' by ∂ and Γ respectively. From (1.11), we have

$$[L_{p,q}, L_{p',q'}] = (qp' - pq')L_{p+q,p'+q'}, \quad [\partial_1, L_{p,q}] = pL_{p,q}, \quad [\partial_2, L_{p,q}] = qL_{p,q},$$

for (p, q), $(p', q') \in \Gamma\backslash\{0\}$. The proof of this claim will be done by several subclaims.

Subclaim 1) $D_0(\partial) = 0$ for $\partial \in T$.

To prove this, applying D_0 to $[\partial, x] = \partial(\beta)x$ for $x \in \mathcal{L}(\Gamma)_\beta$, $\beta \in \Gamma$, as in (2.4), we obtain that $x \cdot D_0(\partial) = 0$. Thus by lemma 2.2, $D_0(\partial) = 0$.

Subclaim 2) By replacing D_0 by $D_0 - u_{\mathrm{inn}}$ for some $u \in V_0$, we can suppose $D_0(L_{p,q}) = 0$ for $p, q, p+q \in \{-1, 0, 1\}$.

We shall simplify notions by denoting

$$L^{p,q}_{r,s} = L_{p,q} \otimes L_{r,s}, \quad L^{(i)}_{p,q} = \partial_i \otimes L_{p,q}, \quad R^{(i)}_{p,q} = L_{p,q} \otimes \partial_i \quad \text{for} \quad (p,q), (r,s) \in \Gamma,\ i = 1, 2.$$

Denote by $\mathrm{Re}\, q$ the real part of q for $q \in \mathbb{C}$. Write

$$D_0(L_{0,1}) = \sum_{p,q} c_{p,q} L^{p,q}_{-p,1-q} + c_1 L^{(1)}_{0,1} + d_1 R^{(1)}_{0,1} + c_2 L^{(2)}_{0,1} + d_2 R^{(2)}_{0,1}, \tag{2.5}$$

for some $c_{p,q}, c_i, d_i \in \mathbb{C}$, where $\{(p, q) \in \Gamma \mid c_{p,q} \neq 0\}$ is a finite set. Note that

$$\begin{aligned}
(L^{p,q-1}_{-p,1-q})_{\mathrm{inn}}(L_{0,1}) &= p(L^{p,q}_{-p,1-q} - L^{p,q-1}_{-p,2-q}),\\
(\partial_2 \otimes \partial_2)_{\mathrm{inn}}(L_{0,1}) &= -R^{(2)}_{0,1} - L^{(2)}_{0,1},\\
(\partial_1 \otimes \partial_2)_{\mathrm{inn}}(L_{0,1}) &= -L^{(1)}_{0,1},\\
(\partial_2 \otimes \partial_1)_{\mathrm{inn}}(L_{0,1}) &= -R^{(1)}_{0,1}.
\end{aligned}$$

Using the above equations, by replacing D_0 by $D_0 - u_{\rm inn}$, where u is a combination of some $L^{p,q-1}_{-p,1-q}, \partial_2\otimes\partial_2, \partial_1\otimes\partial_2, \partial_2\otimes\partial_1$, we can rewrite (2.5) as (recall that $L_{0,0}=0$)

$$D_0(L_{0,1}) = \sum_{q\neq 0,1} c_q L^{0,q}_{0,1-q} + \sum_{p\neq 0,\, 0\leq {\rm Re}\, q<1} c_{p,q} L^{p,q}_{-p,1-q} + cR^{(2)}_{0,1}, \tag{2.6}$$

for some $c_q, c_{p,q}, c\in\mathbb{C}$, where $\{(0,q),(p,q)\in\Gamma \mid c_q, c_{p,q}\neq 0\}$ is a finite set. Write

$$D_0(L_{0,-1}) = \sum_{q\neq 0,1} d_q L^{0,q-1}_{0,-q} + \sum_{p\neq 0} d_{p,q} L^{p,q-1}_{-p,-q} + b_1 L^{(1)}_{0,-1} + f_1 R^{(1)}_{0,-1} + b_2 L^{(2)}_{0,-1} + f_2 R^{(2)}_{0,-1}, \tag{2.7}$$

for some $d_q, d_{p,q}, b_i, f_i\in\mathbb{C}$, where $\{(0,q),(p,q)\in\Gamma \mid d_q, d_{p,q}\neq 0\}$ is a finite set. Applying D_0 to $[L_{0,1}, L_{0,-1}]=0$, we have

$$\sum_{p\neq 0} d_{p,q}(pL^{p,q}_{-p,-q} - pL^{p,q-1}_{-p,1-q}) - b_2 L^{0,1}_{0,-1} - f_2 L^{0,-1}_{0,1} = \sum_{p\neq 0,\, 0\leq {\rm Re}\, q<1} c_{p,q}(pL^{p,q}_{-p,-q} - pL^{p,q-1}_{-p,1-q}) + cL^{0,1}_{0,-1}.$$

Comparing the coefficients, we obtain $d_{p,q} = c_{p,q}$ for $0\leq {\rm Re}\, q<1$, $p\neq 0$, and $d_{p,q} = 0$ for $p\neq 0$, ${\rm Re}\, q<0$ or ${\rm Re}\, q\geq 1$, and $b_2=-c$, $f_2=0$. Thus we can rewrite (2.7) as

$$D_0(L_{0,-1}) = \sum_{q\neq 0,1} d_q L^{0,q-1}_{0,-q} + \sum_{p\neq 0,\, 0\leq {\rm Re}\, q<1} c_{p,q} L^{p,q-1}_{-p,-q} + b_1 L^{(1)}_{0,-1} + f_1 R^{(1)}_{0,-1} - cL^{(2)}_{0,-1}. \tag{2.8}$$

Write

$$D_0(L_{1,0}) = \sum_{p,q} e_{p,q} L^{p+1,q}_{-p,-q} + e_1 L^{(1)}_{1,0} + e_2 R^{(2)}_{1,0} + e'_1 R^{(1)}_{1,0} + e'_2 L^{(2)}_{1,0}, \tag{2.9}$$

for some $e_{p,q}, e_i, e'_i\in\mathbb{C}$, where $\{(p,q)\in\Gamma \mid e_{p,q}\neq 0\}$ is a finite set. Note that

$$\begin{aligned}(L^{0,p}_{0,-p})_{\rm inn}(L_{1,0}) &= -p(L^{1,p}_{0,-p} - L^{0,p}_{1,-p}),\\ (\partial_1\otimes\partial_1)_{\rm inn}(L_{1,0}) &= -R^{(1)}_{1,0} - L^{(1)}_{1,0}.\end{aligned}$$

Using these two equations, by replacing D_0 by $D_0 - u_{\rm inn}$, where u is a combination of some $L^{0,p}_{0,-p}$, $\partial_1\otimes\partial_1$ (this replacement does not affect the above equations (2.6), (2.8)), we can rewrite (2.9) as

$$D_0(L_{1,0}) = \sum_{p\neq 0} e_{p,q} L^{p+1,q}_{-p,-q} + e_1 R^{(1)}_{1,0} + e_2 R^{(2)}_{1,0} + e'_2 L^{(2)}_{1,0}. \tag{2.10}$$

Applying D_0 to $[L_{0,-1},[L_{0,1},L_{1,0}]]=-L_{1,0}$, we have

$$\sum_{p\neq 0}e_{p,q}\Big(-(p+1)^2L^{p+1,q}_{-p,-q}+p(p+1)L^{p+1,q+1}_{-p,-1-q}+p(p+1)L^{p+1,q-1}_{-p,1-q}-p^2L^{p+1,q}_{-p,-q}\Big)-e_1R^{(1)}_{1,0}-e_2R^{(2)}_{1,0}$$
$$+e_2L^{1,1}_{0,-1}+e_2L^{1,-1}_{0,1}+e'_2L^{0,1}_{1,-1}+e'_2L^{0,-1}_{1,1}-e'_2L^{(2)}_{1,0}-\sum_{q\neq 0,1}c_q\Big(qL^{1,q-1}_{0,1-q}-(q-1)L^{0,q}_{1,-q}\Big)-cR^{(2)}_{1,0}$$
$$-\sum_{p\neq 0,\,0\leq \mathrm{Re}\,q<1}c_{p,q}\Big(q(p+1)L^{p+1,q-1}_{-p,1-q}-pqL^{1+p,q}_{-p,-q}-p(q-1)L^{p,q-1}_{1-p,1-q}+(q-1)(p-1)L^{p,q}_{1-p,-q}\Big)$$
$$-\sum_{q\neq 0,1}d_q\Big(-(q-1)L^{1,q}_{0,-q}+qL^{0,q-1}_{1,1-q}\Big)-\sum_{p\neq 0,\,0\leq \mathrm{Re}\,q<1}c_{p,q}\Big((p-q+1)L^{p+1,q}_{-p,-q}-(p-q)L^{p,q-1}_{1-p,1-q}\Big)$$
$$+b_1L^{1,1}_{0,-1}-b_1L^{(1)}_{1,0}-f_1R^{(1)}_{1,0}+f_1L^{0,-1}_{1,1}+cL^{(2)}_{1,0}$$
$$=-\sum_{p\neq 0}e_{p,q}L^{p+1,q}_{-p,-q}-e_1R^{(1)}_{1,0}-e_2R^{(2)}_{1,0}-e'_2L^{(2)}_{1,0}.$$

Comparing the coefficients of $R^{(1)}_{1,0}$, $L^{(1)}_{1,0}$, $R^{(2)}_{1,0}$, $L^{1,1}_{0,-1}$, $L^{1,-1}_{0,1}$, $L^{0,1}_{1,-1}$, $L^{0,-1}_{1,1}$ respectively, we obtain

$$f_1=b_1=c=0,\quad e_2=2c_2=2d_{-1},\quad e'_2=2d_2=2c_{-1}. \tag{2.11}$$

Comparing the coefficients of $L^{0,q}_{1,-q}$, $L^{1,q}_{0,-q}$ with $q\neq 0,\pm 1$ respectively, we obtain

$$(q-1)c_q=(q+1)d_{q+1},\qquad (q+1)c_{q+1}=(q-1)d_q.$$

Note that $d_q=c_q=0$ for $\mathrm{Re}\,q\gg 0$ or $\mathrm{Re}\,q\ll 0$. Thus the above equation forces

$$c_q=d_q=0\ \text{ for } q\neq 0,1 \tag{2.12}$$

From (2.11) and (2.12), we have $e'_2=0$. Comparing the coefficients of $L^{p+1,q}_{-p,-q}$ with $p\neq 0,-1$, $\mathrm{Re}\,q<0$ or $\mathrm{Re}\,q\geq 1$, we obtain

$$e_{p,q-1}+e_{p,q+1}=2e_{p,q}\quad \text{for}\ \ p\neq 0,-1,\ \ \mathrm{Re}\,q<0 \text{ or } \mathrm{Re}\,q\geq 1. \tag{2.13}$$

Replacing q by $q+n$ in (2.13) for $n\in\mathbb{Z}$, one can solve

$$e_{p,q+n}=\begin{cases} e_{p,q}+n(e_{p,q}-e_{p,q-1}) & \text{if}\ \ n\geq 0 \text{ and } \mathrm{Re}\,q\geq 1,\\ e_{p,q}-n(e_{p,q}-e_{p,q+1}) & \text{if}\ \ n\leq 0 \text{ and } \mathrm{Re}\,q<0,\end{cases}$$

for $p\neq 0,-1$. However, $\{(p,q)\in\Gamma\,|\,e_{p,q}\neq 0\}$ is a finite set. We obtain $e_{p,q}=0$ for $p\neq 0,-1$. Comparing the coefficients of $L^{p+1,q}_{-p,-q}$ with $p\neq 0,-1$, $0\leq \mathrm{Re}\,q<1$, we obtain $(p+1)c_{p,q}=pc_{p+1,q}$. Thus $c_{p,q}=0$ for $p\neq 0$, $0\leq \mathrm{Re}\,q<1$. Now (2.6), (2.8) and (2.10) become

$$D_0(L_{0,1})=0,\quad D_0(L_{0,-1})=0,\quad D_0(L_{1,0})=\sum_{q\neq 0}e_qL^{0,q}_{1,-q}+eR^{(1)}_{1,0}, \tag{2.14}$$

where $e=e_1$, $e_q=e_{-1,q}$. Write

$$D_0(L_{-1,0})=\sum_{p,q}f_{p,q}L^{p,q}_{-1-p,-q}+\tilde f_1R^{(1)}_{-1,0}+\tilde f_2L^{(1)}_{-1,0}+f'_1R^{(2)}_{-1,0}+f'_2L^{(2)}_{-1,0}, \tag{2.15}$$

for some $f_{p,q}, \tilde{f}_i, f'_i \in \mathbb{C}$, where $\{(p,q) \in \Gamma \mid f_{p,q} \neq 0\}$ is a finite set. Applying D_0 to $[L_{0,-1}, [L_{0,1}, L_{-1,0}]] = -L_{-1,0}$, using (2.14), we obtain

$$\begin{aligned}
&\sum_{p,q} f_{p,q}\Big(-p^2 L^{p,q}_{-1-p,-q} + p(p+1)L^{p,q+1}_{-1-p,-1-q} + p(p+1)L^{p,q-1}_{-p-1,1-q} - (p+1)^2 L^{p,q}_{-1-p,-q}\Big)\\
&-\tilde{f}_1 R^{(1)}_{-1,0} - \tilde{f}_2 L^{(1)}_{-1,0} - f'_1 R^{(2)}_{-1,0} - f'_1 L^{-1,1}_{0,-1} - f'_1 L^{-1,-1}_{0,1} - f'_2 L^{0,1}_{-1,-1} - f'_2 L^{0,-1}_{-1,1} - f'_2 L^{(2)}_{-1,0}\\
&= -\sum_{p,q} f_{p,q} L^{p,q}_{-1-p,-q} - \tilde{f}_1 R^{(1)}_{-1,0} - \tilde{f}_2 L^{(1)}_{-1,0} - f'_1 R^{(2)}_{-1,0} - f'_2 L^{(2)}_{-1,0}.
\end{aligned}$$

Comparing the coefficients of $L^{0,1}_{-1,-1}$, $L^{-1,-1}_{0,1}$ respectively, we obtain $f'_1 = f'_2 = 0$. Comparing the coefficients of $L^{p,q}_{-1-p,-q}$ with $p \neq 0, -1$, we obtain $f_{p,q-1} + f_{p,q+1} = 2f_{p,q}$. As in (2.13), by noting that $\{(p,q) \in \Gamma \mid f_{p,q} \neq 0\}$ is a finite set, we obtain $f_{p,q} = 0$ for $p \neq 0, -1$. Now we can rewrite (2.15) as

$$D_0(L_{-1,0}) = \sum_{q\neq 0} f_q L^{0,q}_{-1,-q} + \sum_{q\neq 0} f'_q L^{-1,q}_{0,-q} + f R^{(1)}_{-1,0} + g L^{(1)}_{-1,0}, \tag{2.16}$$

where $f_q = f_{0,q}$, $f'_q = f_{-1,q}$, $f = \tilde{f}_1$, $g = \tilde{f}_2$. Applying D_0 to $[L_{-1,0}, L_{1,0}] = 0$, we have

$$\begin{aligned}
&\sum_{q\neq 0} e_q(qL^{-1,q}_{1,-q} - qL^{0,q}_{0,-q}) + eL^{1,0}_{-1,0} + fL^{-1,0}_{1,0} + gL^{1,0}_{-1,0}\\
&= \sum_{q\neq 0} f_q(-qL^{1,q}_{-1,-q} + qL^{0,q}_{0,-q}) + \sum_{q\neq 0} f'_q(-qL^{0,q}_{0,-q} + qL^{-1,q}_{1,-q}).
\end{aligned}$$

Comparing the coefficients of $L^{-1,0}_{1,0}$, $L^{1,0}_{-1,0}$, $L^{1,q}_{-1,-q}$, $L^{-1,q}_{1,q}$ with $q \neq 0$ respectively, we obtain $f = f_q = 0$, $e + g = 0$, $e_q = f'_q$. Thus (2.16) becomes

$$D_0(L_{-1,0}) = \sum_{q\neq 0} e_q L^{-1,q}_{0,-q} - eL^{(1)}_{-1,0}. \tag{2.17}$$

Applying D_0 to $[L_{-1,0}, [L_{1,0}, L_{0,1}]] = -L_{0,1}$, we have

$$\begin{aligned}
&\sum_{q\neq 0} e_q(qL^{-1,q}_{1,1-q} - (q-1)L^{0,q}_{0,1-q}) + eR^{(1)}_{0,1} + eL^{1,1}_{-1,0}\\
&= \sum_{q\neq 0} e_q(-(q+1)L^{0,q+1}_{0,-q} + qL^{-1,q}_{1,1-q}) + eL^{(1)}_{0,1} + eL^{1,1}_{-1,0}.
\end{aligned}$$

Comparing the coefficients of $R^{(1)}_{0,1}$, we obtain $e = 0$. Comparing the coefficients of $L^{0,q}_{0,1-q}$ with $q \neq 0, 1$, we obtain $(q-1)e_q = qe_{q-1}$. Thus $e_q = 0$ for $q \neq 0$, and (2.14), (2.17) become

$$D_0(L_{1,0}) = D_0(L_{-1,0}) = D_0(L_{0,1}) = D_0(L_{0,-1}) = 0. \tag{2.18}$$

From (2.18) we can easily prove Subclaim 2).

Subclaim 3) $D_0(L_{p,q}) = 0$ for $(p,q) \in \Gamma \setminus \{0\}$.

Note that $L_{s,t}$ with $(s,t) \in \mathbb{Z}^2 \setminus \{0\}$ can be generated by $\{L_{p,q} \mid p, q, p+q \in \{0, \pm 1\}\}$. From Subclaim 2), we can easily deduct that $D_0(L_{p,q}) = 0$ for $(p,q) \in \mathbb{Z}^2 \setminus \{0\}$.

For any element $(x,y) \in \Gamma \setminus \mathbb{Z}^2$, write

$$D_0(L_{x,y}) = \sum_{p,q} c_{p,q} L^{p,q}_{x-p,y-q} + a_1 L^{(1)}_{x,y} + b_1 R^{(1)}_{x,y} + a_2 L^{(2)}_{x,y} + b_2 R^{(2)}_{x,y}, \tag{2.19}$$

for some $c_{p,q}, a_i, b_i \in \mathbb{C}$. Applying D_0 to $[L_{0,-1}, [L_{0,1}, L_{x,y}]] = -x^2 L_{x,y}$ and comparing corresponding coefficients, we can obtain $c_{p,q} = 0$ for $p \neq 0, x$. Similarly applying D_0 to $[L_{-1,0}, [L_{1,0}, L_{x,y}]] = -y^2 L_{x,y}$, we have $c_{p,q} = 0$ for $q \neq 0, y$. Thus we can rewrite (2.19) as

$$D_0(L_{x,y}) = e L^{x,0}_{0,y} + f L^{0,y}_{x,0} + a_1 L^{(1)}_{x,y} + b_1 R^{(1)}_{x,y} + a_2 L^{(2)}_{x,y} + b_2 R^{(2)}_{x,y} \quad \text{for } \ e = c_{x,0},\ f = c_{0,y}.$$

Applying D_0 to $[L_{-k,-1}, [L_{k,1}, L_{0,y}]] = -k^2 y^2 L_{0,y}$ and $[L_{-1,-k}, [L_{1,k}, L_{x,0}]] = -k^2 x^2 L_{x,0}$ with $k \gg 0$, we can easily deduct that $D_0(L_{0,y}) = D_0(L_{x,0}) = 0$. Thus now we can assume that $xy \neq 0$. Applying D_0 to $[L_{-k,-1}, [L_{k,1}, L_{x,y}]] = -(x-ky)^2 L_{x,y}$ with $k \gg 0$ such that $x - ky \neq 0$, we have

$$\begin{aligned}
&-ex^2 L^{x,0}_{0,y} + ekxy L^{x+k,1}_{-k,y-1} + ekxy L^{x-k,-1}_{k,y+1} - ek^2y^2 L^{x,0}_{0,y} - fk^2y^2 L^{0,y}_{x,0} + fkxy L^{k,y+1}_{x-k,-1}\\
&+fkxy L^{-k,y-1}_{x+k,1} - fx^2 L^{0,y}_{x,0} + a_1 k(x-ky) L^{k,1}_{x-k,y-1} + a_1 k(x-ky) L^{-k,-1}_{x+k,y+1} - a_1(x-ky)^2 L^{(1)}_{x,y}\\
&-b_1(x-ky)^2 R^{(1)}_{x,y} + b_1 k(x-ky) L^{x+k,y+1}_{-k,-1} + b_1 k(x-ky) L^{x-k,y-1}_{k,1} + a_2(x-ky) L^{k,1}_{x-k,y-1}\\
&+a_2(x-ky) L^{-k,-1}_{x+k,y+1} - a_2(x-ky)^2 L^{(2)}_{x,y} - b_2(x-ky)^2 R^{(2)}_{x,y} + b_2(x-ky) L^{a+k,y+1}_{-k,-1}\\
&+b_2(x-ky) L^{x-k,y-1}_{k,1}\\
&= -(x-ky)^2 (e L^{x,0}_{0,y} + f L^{0,y}_{x,0} + a_1 L^{(1)}_{x,y} + b_1 R^{(1)}_{x,y} + a_2 L^{(2)}_{x,y} + b_2 R^{(2)}_{x,y}).
\end{aligned}$$

Comparing the coefficients of $L^{x,0}_{0,y}, L^{0,y}_{x,0}, L^{k,1}_{x-k,y-1}, L^{x+k,y+1}_{-k,-1}$ respectively, we have

$$ekxy = fkxy = 0, \quad a_1 k + a_2 = b_1 k + b_2 = 0 \tag{2.20}$$

Since $xy \neq 0$ and (2.20) holds for all $k \in \mathbb{Z}$ with $k \gg 0$, we have $e = f = a_1 = a_2 = b_1 = b_2 = 0$. This proves Subclaim 3) and Claim 3.

Claim 4. For every $D \in \mathrm{Der}(\mathcal{L}(\Gamma), V)$, (2.2) is a finite sum.

By Claims 2 and 3, we can suppose $D_\alpha = (v_\alpha)_{\mathrm{inn}}$ for some $v_\alpha \in V_\alpha$ and $\alpha \in \Gamma$. If $\Gamma' = \{\alpha \in \Gamma \backslash \{0\} \mid v_\alpha \neq 0\}$ is an infinite set, by linear algebra, there exists $\partial \in T$ such that $\partial(\alpha) \neq 0$ for $\alpha \in \Gamma'$. Then $D(\partial) = \sum_{\alpha \in \Gamma' \cup \{0\}} \partial \cdot v_\alpha = \sum_{\alpha \in \Gamma'} \partial(\alpha) v_\alpha$ is an infinite sum, thus not an element in V. This is a contradiction with the fact that D is a derivation from $\mathcal{L}(\Gamma) \to V$. This proves Claim 4 and the lemma.

Lemma 2.4. *Suppose $r \in V$ such that $a \cdot r \in \mathrm{Im}(1-\tau)$ for all $a \in \mathcal{L}(\Gamma)$. Then $r \in \mathrm{Im}(1-\tau)$*

Proof. (cf. [SS], [WS]) First note that $\mathcal{L}(\Gamma)\cdot\mathrm{Im}(1-\tau) \subset \mathrm{Im}(1-\tau)$. We shall prove that after a number of steps in each of which r is replaced by $r-u$ for some $u \in \mathrm{Im}(1-\tau)$, the zero element is obtained and thus proving that $r \in \mathrm{Im}(1-\tau)$. Write $r = \sum_{x\in\Gamma} r_x$. Obviously,

$$r \in \mathrm{Im}(1-\tau) \iff r_x \in \mathrm{Im}(1-\tau) \text{ for all } x \in \Gamma. \tag{2.21}$$

For any $x' \neq 0$, choose $\partial \in T$ such that $\partial(x') \neq 0$. Then $\sum_{x\in\Gamma} \partial(x) r_x = \partial \cdot r \in \mathrm{Im}(1-\tau)$. By (2.21), $\partial(x) r_x \in \mathrm{Im}(1-\tau)$, in particular, $r_{x'} \in \mathrm{Im}(1-\tau)$. Thus by replacing r by $r - \sum_{0\neq x\in\Gamma} r_x$, we can suppose $r = r_0 \in V_0$. Now we can write

$$r = \sum_{p,q} c_{p,q} L^{p,q}_{-p,-q} + c_1 \partial_1 \otimes \partial_1 + c_1' \partial_1 \otimes \partial_2 + c_2 \partial_2 \otimes \partial_1 + c_2' \partial_2 \otimes \partial_2, \tag{2.22}$$

for some $c_{p,q}, c_i, c_i' \in \mathbb{C}$. Choose any total order on $\mathbb{C}$ compatible with its group structure. Since $v_{p,q} := L^{p,q}_{-p,-q} - L^{-p,-q}_{p,q} \in \mathrm{Im}(1-\tau)$, by replacing r by $r-u$, where u is a combination of some $v_{p,q}$. We can suppose

$$c_{p,q} \neq 0 \implies p > 0 \text{ or } p = 0, q > 0. \tag{2.23}$$

First assume that $c_{p,q} \neq 0$ for some p, q. Choose $s, t > 0$ such that $sq - pt \neq 0$. Then we see that the term $L^{p+s,q+t}_{-p,-q}$ appears in $L_{s,t} \cdot r$, but (2.23) implies that the term $L^{-p,-q}_{p+s,q+t}$ does not appear in $L_{s,t} \cdot r$, a contradiction with the fact that $L_{s,t} \cdot r \in \mathrm{Im}(1-\tau)$. Now write $r = c_1 \partial_1 \otimes \partial_1 + c_1' \partial_1 \otimes \partial_2 + c_2 \partial_2 \otimes \partial_1 + c_2' \partial_2 \otimes \partial_2$. Then from

$$\begin{aligned} L_{1,0} \cdot r &= -c_1 R^{(1)}_{1,0} - c_1 L^{(1)}_{1,0} - c_1' R^{(2)}_{1,0} - c_2 L^{(2)}_{1,0} \in \mathrm{Im}(1-\tau), \\ L_{0,1} \cdot r &= -c_1' L^{(1)}_{0,1} - c_2 R^{(1)}_{0,1} - c_2' R^{(2)}_{0,1} - c_2' L^{(2)}_{0,1} \in \mathrm{Im}(1-\tau), \end{aligned}$$

we obtain that $c_1 = 0$, $c_1' + c_2 = 0$, $c_2 = 0$. Thus $r \in \mathrm{Im}(1-\tau)$. This proves the lemma.

Proof of Theorem 1.3(1). Let $(\mathcal{L}(\Gamma), [\cdot,\cdot], \Delta)$ be a Lie bialgebra structure on $\mathcal{L}(\Gamma)$. By (1.6), (1.13) and Theorem 1.3(3), $\Delta = \Delta_r$ is defined by (1.8) for some $r \in \mathcal{L}(\Gamma) \otimes \mathcal{L}(\Gamma)$. By (1.4), $\mathrm{Im}\,\Delta \subset \mathrm{Im}(1-\tau)$. Thus by Lemma 2.4, $r \in \mathrm{Im}(1-\tau)$. Then by (1.5), (2.1) and Theorem 1.3(2) show that $c(r) = 0$. Thus Definition 1.2 says that $(\mathcal{L}(\Gamma), [\cdot,\cdot], \Delta)$ is a triangular coboundary Lie bialgebra. □

References

[D1] V.G. Drinfeld, Hamlitonian structures on Lie group,Lie algebras and the geometric meaning of classical Yang-Baxter equations, *Soviet Math. Dokl.* **27**(1) (1983), 68-71.

[D2] V.G. Drinfeld, Quantum groups, in: *Proceeding of the International Congress of Mathematicians*, Vol. 1, 2, Berkeley, Calif. 1986, Amer. Math. Soc., Providence, RI, 1987, pp. 798-820.

[DZ] D. Dokovic, K. Zhao, Derivations, isomorphisms and second cohomology of generalized Block algebras, *Algebra Colloquium* **3** (1996), 245–272.

[LT] W. Lin, S. Tan, Nonzero level Harish-Chandra modules over the Virasoro-like algebra, *J. Pure Appl. Algebra* (2006), in press.

[MJ] D. Meng, C. Jiang, The derivation algebra and the universal central extension of the q-analog of the Virasoro-like algebra, *Comm. Algebra* **26** (1998), 1335–1346.

[M1] W. Michaelis, A class of infinite-dimensional Lie bialgebras containing the Virasoro algebras, *Adv. Math.* **107** (1994), 365–392.

[M2] W. Michaelis, The dual Poincare-Birkhoff-Witt theorem, *Adv. Math.* **57** (1985), 93–162.

[NT] S.-H. Ng, E.J. Taft, Classification of the Lie bialgebra structures on the Witt and Virasoro algebras, *J. Pure Appl. Algebra* **151** (2000), 67–88.

[N] W.D. Nichols, The structure of the dual Lie coalgebra of the Witt algebra, *J. Pure Appl. Algebra* **68** (1990), 395–364.

[SS] G. Song, Y. Su, Lie bialgebras of generalized Witt type, *Science in China A*, in press (see arXiv:math.QA/0504168).

[SX] Y. Su, X. Xu, Structure of divergence-free Lie algebras, *J. Algebra* **243** (2001), 557–595.

[T] E.J. Taft, Witt and Virasoro algebras as Lie bialgebras, *J. Pure Appl. Algebra* **87** (1993), 301–312.

[WS] Y. Wu, Y. Su, Nongraded Lie bialgebras of generalized Witt type, to appear.

[X1] X. Xu, New generalized simple Lie algebras of Cartan type over a field with characteristic 0, *J. Algebra* **224** (2000), 23–58.

[X2] X. Xu, Novikov-Poisson algebras, *J. Algebra* **190** (1997), 253–279.

[X3] X. Xu, Generalizations of Block algebras, *Manuscripta Math.* **100** (1999), 489–518.

[ZZ] H. Zhang, K. Zhao, Representations of the Virasoro-like algebra and its q-analog, *Comm. Algebra* **24** (1996), 4361–4372.

[Z1] K. Zhao, A class of infinite dimensional simple Lie algebras, *J. London Math. Soc. (2)*, **62** (2000), 71–84.

[Z2] K. Zhao, Generalized Cartan type S Lie algebras in characteristic zero, II, *Pacific J. Math.* **192** (2000), 431–454.

[ZM] L. Zhu, D. Meng, Structure of degenerate Block algebras, *Algebra Colloquium* **10** (2003), 53–62.